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Block diagonalizing ultrametric matrices

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Abstract. The problem of diagonalizing a class of complicated matrices, to be called ultrametric matrices, is investigated. These matrices appear at various stages in the description of disordered systems with many equilibrium phases by the technique of replica-symmetry breaking. The residual symmetry, remaining after the breaking of permutation symmetry between replicas, allows us to bring all ultrametric matrices to a block diagonal form by a common similarity transformation. A large number of these blocks are, in fact, of size 1×1 , i.e. in a vast sector the transformation actually diagonalizes the matrix. In the other sectors we end up with blocks of size $(R + 1) \times (R + 1)$, where R is the number of replica-symmetry-breaking steps. These blocks cannot be further reduced without giving more information, in addition to ultrametric symmetry, about the matrix. Simialr results for the inverse of a generic ultrametric matrix are also derived.

1. Introduction

Low-temperature disordered systems often possess many equilibrium phases. The technique of replica-symmetry breaking (RSB) provides a theoretical framework in which these systems can be described analytically, starting from a microscopic basis. Discovered and developed in the theory of spin glasses [1], RSB has recently penetrated into a number of other problems, including the theory of random manifolds [2–4], random-field problems [5, 6], protein folding [7–9], vortex pinning [10], etc. In each of these theories randomness is handled via the replica trick, and the multitude of equilibrium phases is captured by breaking the permutation symmetry between the replicas. As always, symmetry breaking means that the low-temperature solutions realize a particular subgroup of the underlying symmetry group of the theory, here of the permutation group. The proper choice of the subgroup proved to be a highly non-trivial task in the case of RSB. The succesful ansatz for the symmetry breaking pattern, proposed by Parisi originally in the context of spin glasses, turned out to embody a particular hierarchical organization of the equilibrium phases, usually referred to as ultrametricity [1].

The corresponding subgroup determines the structure not only of the order parameter, but also of all other quantities in the theory, like self-energies, propagators, etc. The structure imposed by this residual symmetry on quantities depending on two replica indices is by now widely known. The algebra of such quantities has been worked out by Parisi [11] with further results, most notably on the inversion problem, added by Mézard and Parisi [2]. At a certain stage of the development of RSB theories, however, one also has to

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face more complicated objects, depending on three or four replica indices. The structure of these is much harder to grasp and their algebra is much more involved than that of the two-index quantities. Our purpose here is to analyse and exploit the structure imposed by ultrametricity on four-index quantities. For reasons to become clear shortly, we shall call them ultrametric matrices and will be concerned, in particular, with their diagonalization and inversion.

In order to keep the full generality of the results and thereby guarantee their applicability in any RSB theory, we shall not assume any properties other than those imposed by ultrametricity on these matrices. In this way we separate the analysis of the purely geometric aspects of RSB theories, which are common to all of them, from the treatment of other properties which are determined by more specific details of the particular systems.

Also, we shall keep the number n of replicas a positive integer throughout this paper. The replica limit $n \rightarrow 0$ is, of course, the most essential step of the replica method. It is also the source of mathematical ambiguities. The analysis of the consequences of ultrametric symmetry, however, does not depend on n , therefore we found it useful to keep it finite. This way our analysis belongs to the realm of well established mathematics and the analytic continuation in n can be carried out at the latest stage, on the final results.

The number R of replica-symmetry-breaking steps will also be considered a generic integer. Thus our results will be applicable in situations where only a single RSB step is needed, as well as in the case of fully fledged RSB with $R \rightarrow \infty$. The results for this 'continuous' case ($R \rightarrow \infty, n \rightarrow 0$), derived by a completely different method, have been published in [12].

Although almost trivial in principle, our analysis will, inevitably, be very complicated in actual details. It is clearly impossible to reproduce the often very lengthy calculations here, and we shall have to use the phrase 'it can be shown' frequently. What we mean at these points is that one can reproduce the results more easily than follow the lengthy proofs. A good strategy is to work out a simple special case (like that with $R = 1$) first; the induction is easy to spot in most cases.

The plan of the paper is the following. In section 2 the definition of ultrametric matrices is presented together with the classification of their different matrix elements. Section 3 contains a detailed analysis of the non-orthogonal basis vectors of a similarity transformation which brings all ultrametric matrices to a block-diagonal form. In section 4 this block-diagonal form is expressed through some 'kernels', which facilitate the eigenvalue and inversion problem greatly. A complete list of matrix components versus kernels is also included in this section. Some technicalities are relegated to the appendix.

2. Definition of a generic ultrametric matrix

For the sake of definiteness we present our analysis in the language of spin glasses, the extension to other replica-symmetry-breaking theories is merely a matter of notation. The replica method yields the free-energy F of a long-range spin glass in the form of a functional, depending on a set $q_{\alpha\beta}$ of order parameters: $F = F(q_{\alpha\beta})$. The replica indices α, β take integer values: $\alpha, \beta = 1, 2, \dots, n$ (for our present analysis the replica limit $n \rightarrow 0$ need not be considered here). The order parameters are symmetric: $q_{\alpha\beta} = q_{\beta\alpha}$, and (for Ising spins) the diagonal components are zero: $q_{\alpha\alpha} = 0$. The number of independent order parameter components is thus $\frac{1}{2}n(n-1)$. The free energy is independent of the labelling

of the replicas, so F must be constructed from the algebraic invariants of the permutation group of n objects. Examples of such invariant combinations are

$$\begin{aligned} \sum_{\alpha\beta} q_{\alpha\beta} & \quad \text{Tr } q^2 = \sum_{\alpha\beta} q_{\alpha\beta}^2 & \quad \text{Tr } q^3 = \sum_{\alpha\beta\gamma} q_{\alpha\beta} q_{\beta\gamma} q_{\gamma\alpha} \\ \sum_{\alpha\beta} q_{\alpha\beta}^4 & \quad \text{Tr } q^4 = \sum_{\alpha\beta\gamma} q_{\alpha\beta}^2 q_{\beta\gamma}^2 & \quad \sum_{\alpha\beta\gamma\delta} q_{\alpha\beta}^2 q_{\alpha\gamma} q_{\gamma\delta} q_{\delta\alpha} & \quad \text{etc.} \end{aligned}$$

The stationary values of the order-parameter components are determined by the equation of state: $\partial F/\partial q_{\alpha\beta} = 0$. There are $\frac{1}{2}n(n-1)$ such equations. Depending on the parameters in F these equations may have solutions that are indentially zero, $q_{\alpha\beta} = 0, \forall \alpha, \beta$, or that have non-zero but identical off-diagonal components $q_{\alpha\beta} = q(1 - \delta_{\alpha,\beta}^{Kr})$, or non-zero off-diagonal components that depend on the pair (α, β) of replica indices. The solutions of the last kind are said to be replica-symmetry-breaking (RSB) solutions and these are the ones that describe situations where many equilibrium states exist. Led by a number of formal considerations which later turned out to embody the ultrametric organization of these states, Parisi proposed a, by now standard, parametrization for the RSB solutions which we briefly recapitulate in order to fix notations.

Firstly assume that n is not only an integer, but a very large one, with a large number of proper divisors. Let p_1 be one of these, itself a large number with many divisors, let one of them be p_2 , etc, up to p_R . It is useful to rename n as p_0 , and add $p_{R+1} \equiv 1$ to the other end of the series. Now the n replicas are divided into n/p_1 boxes each containing p_1 replicas. The contents of each box are further subdivided into p_1/p_2 smaller boxes with p_2 replicas in each etc, down to the smallest boxes with p_R replicas. The RSB solutions are supposed to be invariant WRT the permutations of replicas inside each of the smallest boxes of size p_R , and also WRT the permutations of the size p_{k+1} boxes inside each of the size p_k boxes for any $k = 0, 1, \dots, R-1$. Evidently, these permutations form a subgroup of the permutation group. This subgroup is the residual symmetry that remains after the breaking of replica symmetry. The ansatz for the order-parameter matrix corresponding to this residual symmetry is constructed as follows: the $n \times n$ (i.e. $p_0 \times p_0$) matrix $q_{\alpha\beta}$ is divided into blocks of size $p_1 \times p_1$, and a common value q_0 is assigned to all matrix elements outside the diagonal blocks. Next, the diagonal blocks are further divided into blocks of size $p_2 \times p_2$, the value q_1 assigned to the matrix elements inside the diagonal blocks of size $p_1 \times p_1$ but outside the diagonal blocks of size $p_2 \times p_2$, etc, down to the innermost blocks of size $p_R \times p_R$, where the matrix elements are q_R except along the diagonal of the whole matrix where they are zero. Some formulae below (equations (32) and (34)) become meaningless whenever the ratio of two subsequent p 's is 2 or 3. These cases would require a separate discussion which we can safely omit here, since in practical applications these cases will never appear. For our present purposes we can stipulate $p_k/p_{k+1} > 3, k = 0, 1, \dots, R$.

The solution of the stationary condition $\partial F/\partial q_{\alpha\beta} = 0$ is sought among the matrices which have the special form just described. This solution is a point in the $\frac{1}{2}n(n-1)$ -dimensional replica space, so it is, in fact, a *vector*. In the following, when we deal with genuine matrices acting on replica space, i.e. with quantities depending on two pairs of replica indices, we will actually call $q_{\alpha\beta}$ and similar quantities vectors. The association between the $n \times n$ symmetric matrix $q_{\alpha\beta}$ (with $q_{\alpha\alpha} = 0$) and the column vector $|q_{\alpha\beta}\rangle$ is evident: one lists the matrix elements above the diagonal of $q_{\alpha\beta}$ in any prescribed order (say, row by row) below each other.

The representation of $q_{\alpha\beta}$ and other vectors of replica space by symmetric matrices remains, nevertheless, very useful, because it is much easier to display their special

structure in the matrix form. Therefore, we shall use this matrix representation for all vectors appearing in this paper. For later reference we note here that the scalar product of two vectors, $|r_{\alpha\beta}\rangle$ and $|q_{\alpha\beta}\rangle$, is, in matrix language, half the trace of the product of the corresponding matrices:

$$\langle r_{\alpha\beta} | q_{\alpha\beta} \rangle = \frac{1}{2} \text{Tr}(rq). \quad (1)$$

We now introduce the concept of the overlap between replica indices that will play a central role in the following: the overlap between α and β is k (notation: $\alpha \cap \beta = k$) if in the Parisi scheme $q_{\alpha\beta} = q_k$. The overlap $\alpha \cap \beta$ defined in this way can be regarded as a kind of hierarchical distance between replicas α and β , its values ranging from 0 (corresponding to the largest off-diagonal blocks of size $p_1 \times p_1$) to $R + 1$ (corresponding to the diagonal, $\alpha = \beta$).

It is evident that any quantity f constructed of the q 's and depending on only two replica indices (such as $f_{\alpha\beta} = \sum_{\gamma} q_{\alpha\gamma} q_{\gamma\beta}$, for example) only depends on their overlap: $f_{\alpha\beta} = f(\alpha \cap \beta)$.

The metric generated by the overlaps is, by construction, ultrametric: whichever way we choose three replicas α, β, γ , either all three of their overlaps are the same ($\alpha \cap \beta = \alpha \cap \gamma = \beta \cap \gamma$), or one (say $\alpha \cap \beta$) is larger than the other two, but then these are equal ($\alpha \cap \beta > \alpha \cap \gamma = \beta \cap \gamma$).

Furthermore, it also follows that any quantity f built of the q 's and depending on three replica indices, $f_{\alpha\beta\gamma}$, depends only on the overlaps $\alpha \cap \beta$, $\alpha \cap \gamma$, $\beta \cap \gamma$, and since of these at most two can be different, $f_{\alpha\beta\gamma}$ is, in fact, a function of only two variables, e.g. of $\alpha \cap \beta$ and of the larger of the other two:

$$f_{\alpha\beta\gamma} = f(\alpha \cap \beta; \max\{\alpha \cap \gamma, \beta \cap \gamma\}). \quad (2)$$

In the following we will also have to consider quantities depending on four replica indices, coming in two pairs: $f_{\alpha\beta,\gamma\delta}$. A little reflection shows that such a quantity can always be parametrized as follows:

$$f_{\alpha\beta,\gamma\delta} = f_{\max\{\alpha \cap \gamma, \alpha \cap \delta\}, \max\{\beta \cap \gamma, \beta \cap \delta\}}^{\alpha \cap \beta, \gamma \cap \delta}. \quad (3)$$

Admittedly, this parametrization is less than perfect. Firstly, ultrametricity implies that of the six possible overlaps between α, β, γ and δ at most three can be different, which corresponds to the simple geometric fact that the edges of a tetrahedron having equilateral or isosceles faces can have at most three different lengths. Therefore, of the four variables on the RHS of (3) at least two are the same. The resulting redundancy is the price we pay for the symmetry of the notation. Secondly, in all practical applications $f_{\alpha\beta,\gamma\delta}$ is symmetric WRT exchanging α and β or γ and δ and also WRT exchanging the two pairs: $f_{\alpha\beta,\gamma\delta} = f_{\alpha\beta,\delta\gamma} = f_{\beta\alpha,\gamma\delta} = f_{\gamma\delta,\alpha\beta}$ etc, and these symmetries are not manifestly reflected by the parametrization (3). We prefer keeping the consequences of these symmetries in mind rather than overcomplicating the notation.

The choice between the various types of solutions of the equation of state (identically zero, or constant $q_{\alpha\beta}$, or replica-symmetry broken $q_{\alpha\beta}$) is based on stability considerations. In order to decide the stability of a given solution, one has to diagonalize the Hessian or (bare) self-energy matrix $\partial^2 F / \partial q_{\alpha\beta} \partial q_{\gamma\delta} = M_{\alpha\beta,\gamma\delta}$, evaluated at the stationary point. M is the prime example of a quantity depending on two pairs of replica indices, so it can be parametrized as shown in (3). M is obviously symmetric WRT the exchange of the two pairs $(\alpha\beta)$ and $(\gamma\delta)$. Since $q_{\alpha\beta} = q_{\beta\alpha}$ and $q_{\alpha\alpha} = 0$, M can be considered to depend on the

ordered pairs $\alpha < \beta$ and $\gamma < \delta$ only, so it is a matrix of dimension $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$. A symmetric matrix of this size has, in general, $\frac{1}{4}n(n-1)[\frac{1}{2}n(n-1) + 1]$ independent elements. This number is greatly reduced by ultrametricity. Below we list all the different kinds of matrix elements which can appear. When doing so, we will relax the ordering of the indices of $M_{\alpha\beta,\gamma\delta}$, and extend the definition to arbitrary combinations of the indices (except $\alpha = \beta$ and $\gamma = \delta$) so as to make M symmetric WRT exchanging α and β and/or γ and δ , $M_{\alpha\beta,\gamma\delta} = M_{\alpha\beta,\delta\gamma} = M_{\beta\alpha,\gamma\delta} = M_{\beta\alpha,\delta\gamma}$, in addition to the symmetry WRT exchanging the pairs $(\alpha\beta)$ and $(\gamma\delta)$. This extension is motivated by convenience: when summations are to be performed on the indices of M the restrictions due to ordering can become very cumbersome.

The matrix elements can be classified naturally in three categories.

(i) *Matrix elements of the first kind.* These are the diagonal elements $M_{\alpha\beta,\alpha\beta}$, together with their variants $M_{\alpha\beta,\beta\alpha}$, $M_{\beta\alpha,\alpha\beta}$, etc. They depend on the overlap $\alpha \cap \beta = i = 0, 1, 2, \dots, R$ only. Under the parametrization (3) they are given by

$$M_{\alpha\beta,\alpha\beta} = M_{R+1,R+1}^{i,i} \quad i = 0, 1, \dots, R. \quad (4)$$

There are, in general, $R+1$ different matrix elements in this category (instead of $\frac{1}{2}n(n-1)$, the dimension of the matrix).

(ii) *Matrix elements of the second kind.* These are off-diagonal, with one replica index in common between the two pairs. One example is $M_{\alpha\beta,\alpha\gamma}$ which, together with its exchanged variants ($M_{\alpha\beta,\gamma\alpha}$ etc), exhausts all possibilities.

There are three cases:

(a) $\alpha \cap \beta = \alpha \cap \gamma = i \leq \beta \cap \gamma = j = 0, 1, \dots, R$. Then

$$M_{\alpha\beta,\alpha\gamma} = M_{R+1,j}^{i,i} \quad j \geq i. \quad (5)$$

Various exchanges of the replica indices either reproduce the same, or exchange the lower variables $R+1$ and j . (The parametrization (3) is such that $M_{k,l}^{i,j}$ is always symmetric in k and l .)

(b) $\alpha \cap \beta = \beta \cap \gamma = i < \alpha \cap \gamma = j$.

$$M_{\alpha\beta,\alpha\gamma} = M_{R+1,i}^{i,j} \quad j > i. \quad (6)$$

Exchanging replica indices in all possible ways reproduces either the same, or exchanges the lower variables, or exchanges i and j . Thus

$$M_{R+1,i}^{i,j} = M_{i,R+1}^{i,j} = M_{R+1,j}^{j,i} = M_{j,R+1}^{j,i} \quad i < j. \quad (7)$$

(c) $\alpha \cap \beta = i > \alpha \cap \gamma = \beta \cap \gamma = j$.

$$M_{\alpha\beta,\alpha\gamma} = M_{R+1,i}^{i,j} \quad i > j.$$

According to (7), this is the same as (6) (rename $i \leftrightarrow j$).

It is easy to see that the number of different matrix elements is at most $(R+1)^2$ in this class.

(iii) *Matrix elements of the third kind.* These have four different replica indices, $M_{\alpha\beta,\gamma\delta}$. Considering all logically possible situations with $\alpha < \beta$, $\gamma < \delta$, $\alpha < \gamma$, $\beta < \delta$

(corresponding to the matrix elements above the diagonal of M), we find six possible cases altogether.

(a) $\alpha \cap \beta = i$, $\gamma \cap \delta = j$, $\max\{\alpha \cap \gamma, \alpha \cap \delta\} = \max\{\beta \cap \gamma, \beta \cap \delta\} = k$ with $k \leq \min\{i, j\}$.

Then

$$M_{\alpha\beta,\gamma\delta} = M_{k,k}^{i,j} = M_{k,k}^{j,i} \quad (8)$$

where the second equality follows from exchanging the two pairs $(\alpha\beta) \leftrightarrow (\gamma\delta)$.

(b) $\alpha \cap \beta = i$, $\gamma \cap \delta = j$, $\max\{\alpha \cap \gamma, \alpha \cap \delta\} = \max\{\beta \cap \gamma, \beta \cap \delta\} = k$, $j < k \leq i$. Then

$$M_{\alpha\beta,\gamma\delta} = M_{k,k}^{i,j} = M_{k,j}^{j,i} \quad (9)$$

(c) $\alpha \cap \beta = i$, $\gamma \cap \delta = j$, $\max\{\alpha \cap \gamma, \alpha \cap \delta\} = i$, $\max\{\beta \cap \gamma, \beta \cap \delta\} = k$, $j \leq i < k$. Then

$$M_{\alpha\beta,\gamma\delta} = M_{i,k}^{i,j} = M_{k,j}^{j,i} = M_{j,k}^{j,i} \quad (10)$$

(d) $\alpha \cap \beta = i$, $\gamma \cap \delta = j$, $\max\{\alpha \cap \gamma, \alpha \cap \delta\} = i$, $\max\{\beta \cap \gamma, \beta \cap \delta\} = k$, $i < j \leq k$. Then

$$M_{\alpha\beta,\gamma\delta} = M_{i,k}^{i,j} = M_{k,j}^{j,i} \quad (11)$$

(e) $\alpha \cap \beta = i$, $\gamma \cap \delta = j$, $\max\{\alpha \cap \gamma, \alpha \cap \delta\} = i$, $\max\{\beta \cap \gamma, \beta \cap \delta\} = k$, $i < k < j$. Then

$$M_{\alpha\beta,\gamma\delta} = M_{i,k}^{i,j} = M_{k,k}^{j,i} \quad (12)$$

(f) $\alpha \cap \beta = \gamma \cap \delta = i$, $\max\{\alpha \cap \gamma, \alpha \cap \delta\} = k$, $\max\{\beta \cap \gamma, \beta \cap \delta\} = l$, $i < \min\{k, l\}$. Then

$$M_{\alpha\beta,\gamma\delta} = M_{k,l}^{i,i} \quad (13)$$

In equations (8)–(13) the overlaps i, j, k, l can run through $0, 1, 2, \dots, R$. Considering the cases (a)–(f) one can show that the number of different matrix elements of the third kind is $(R+1)^3$.

If all the $\frac{1}{2}n(n-1)$ independent order-parameter components were different, the matrix $M_{\alpha\beta,\gamma\delta}$ would have of the order of n^4 independent matrix elements. Parisi's RSB scheme does not completely destroy the permutation symmetry of the replicas, however, it only reduces this symmetry to a particular subgroup of the group of permutations of n elements. It is this residual symmetry which is responsible for the tremendous reduction in the number of independent elements of the Hessian: instead of $O(n^4)$ we have, according to equations (4)–(13), only $O(R^3)$ different matrix elements, which, for large n , is exponentially small compared to n^4 .

The particular structure described above has been displayed in the example of the Hessian of the long-range spin glass. Matrices with an identical structure appear in many RSB theories. We shall call these matrices ultrametric matrices. From this point on we shall disregard the derivation and meaning of M , and will focus solely on its symmetries. It will be seen that these symmetries allow one to construct an irreducible representation for ultrametric matrices in that all those which have the same block sizes p_0, p_1, \dots, p_R can be brought to a block-diagonal form by the same similarity transformation and that no further reduction is possible without providing further information on the matrix elements. It will also be seen that the conditions ultrametricity imposes upon M are stringent enough to actually yield a large number of the eigenvalues in closed form. We shall also look into the problem of inversion of ultrametric matrices and shall find again that a large number of the

components of the inverse can be obtained in closed form. Some of the results we compile here are not new—they were published by two of us some years ago in a very compact form [13]. In addition to re-phrasing them and providing some background material, we also present a number of new results, especially with regard to the inversion problem. As long as n is an integer with the sequence of divisors p_1, p_2, \dots, p_R as described, the matrix M is a well defined mathematical object, and the problem of its diagonalization belongs to the realm of standard mathematics. The present paper will be concerned with this well posed problem. At a certain point one will, however, have to consider the replica limit $n \rightarrow 0$, together with the analytic continuation in all the p_i 's and with the limit $R \rightarrow \infty$, as proposed by Parisi [1]. These manipulations are, at the present time, of a purely formal character, certainly beyond the limits of well established mathematics. After all these dubious steps one arrives at the problem of the diagonalization of an integral operator with a set of particular symmetries. The results we get in the discrete case can all be easily transcribed onto this new, continuous problem.

3. The new basis

In the previous section the components of an ultrametric matrix $M_{\alpha\beta,\gamma\delta}$ have been given in the Cartesian coordinate system spanned by the basis vectors $|\mu, \nu\rangle, (\mu, \nu) = (1, 2), (1, 3), \dots, (n-1, n)$, which, similarly to the order parameter $q_{\alpha\beta}$, can be represented by symmetric $n \times n$ matrices. Their matrix elements are

$$|\mu, \nu\rangle_{\alpha\beta} = \delta_{(\mu,\nu),(\alpha,\beta)}^{Kr} = \begin{cases} 1 & \text{if } \mu = \alpha, \nu = \beta \text{ or } \mu = \beta, \nu = \alpha \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

In order to bring M to a block-diagonal form, we have to go over into a new basis. The new set of basis vectors can be inferred from the general structure of the eigenvectors described in [13] and, like those, can be naturally classified in three families.

The first family.

The first family of basis vectors consists of $R + 1$ vectors labelled by $i = 0, 1, \dots, R$ which, when represented by quadratic matrices, have identical, non-zero elements on the i th level of the Parisi hierarchy and zeros everywhere else:

$$|0; i\rangle = \left(\frac{1}{2}n(p_i - p_{i+1})\right)^{-1/2} \sum_{\alpha \cap \beta = i} |\alpha, \beta\rangle \tag{15}$$

where the $|\alpha, \beta\rangle$'s are the Cartesian unit vectors defined in (14). The meaning of the first label (0) will become clear shortly.

The first family basis vectors form an $(R + 1)$ -dimensional orthonormal set in replica space. The difference $p_i - p_{i+1}$ in the normalization factor will appear so often in the following that it is worth giving it a name:

$$p_i - p_{i+1} = \delta_i \quad i = 0, 1, \dots, R. \tag{16}$$

A straightforward but tedious calculation shows that the subspace spanned by the first family basis vectors is closed under the action of an ultrametric matrix. Therefore the linear combination

$$|f\rangle = \sum_{i=0}^R f_0(i) |0; i\rangle \tag{17}$$

is an eigenvector of M , provided the amplitudes $f_0(i)$ are appropriately chosen. The conditions for these amplitudes (i.e. the eigenvalue equations) will be written up in the next section. Evidently, there will be $R + 1$ possible choices for the amplitudes, corresponding to $R + 1$ eigenvalues $\lambda_m(0)$, $m = 0, 1, \dots, R$. In the case of a generic matrix M with no symmetries other than those dictated by ultrametricity, all these eigenvalues will be singlets, their multiplicity $\mu(0) = 1$, and the eigenvectors orthogonal. In the following we shall often refer to the first family as the longitudinal or L family.

The second family.

The second family will be broken down into several subfamilies, to be labelled by an index $k = 1, 2, \dots, R + 1$. The first family is, in several respects, nothing but the case corresponding to $k = 0$, which is why we used the label 0 in addition to i . The structure of the second family basis vectors is easiest to grasp graphically, so we define them in a series of figures. The vectors belonging to the $k = 1$ subfamily are shown in figures 1 and 2. (Although, as we have already mentioned, the ratios of subsequent p 's must never be 2 or 3 in order to prevent the figures from occupying an excessive space, here and in almost all the figures to follow we have to illustrate the structure of eigenvectors by figures where some of these ratios are 3.) Consider the vectors shown in figure 1. They only have non-zero components on the zeroth level of the Parisi hierarchy, but now not all these components are identical: they take two different values, A and B , arranged as shown in the figure. We shall denote these vectors as $|1; 0; b\rangle$, where the first label is the value of k , the second is that level of the Parisi hierarchy where the vector has non-zero elements, and the third, $b = 1, 2, \dots, n/p_1$, shows which column and row of blocks is distinguished, i.e. which blocks have matrix elements B .

Now consider the sum of these vectors, $\sum_{b=1}^{n/p_1} |1; 0; b\rangle$. The distinction between the different blocks will obviously disappear in the sum, so it will be proportional to $|0; 0\rangle$ of the first family. However, we want to make each of the second family vectors orthogonal to the first family, so we have to choose the vector components A and B so as to make the above sum vanish. If we choose, say, $A = 1$ then B must be

$$B = \frac{1}{2} \left(2 - \frac{n}{p_1} \right). \tag{18}$$

With this choice the $|1; 0; b\rangle$ vectors are all orthogonal to the first family and

$$\sum_{b=1}^{n/p_1} |1; 0; b\rangle = 0. \tag{19}$$

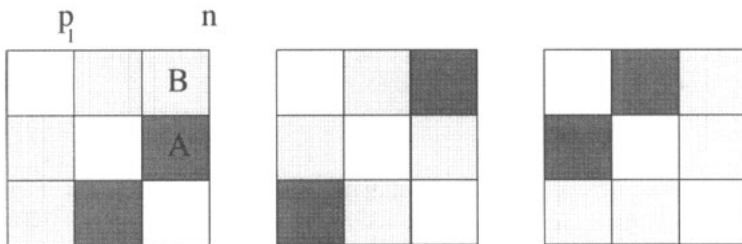


Figure 1. The vectors $|k = 1, i = 0, b\rangle$. The label $b = 1, 2, \dots, n/p_1$ shows which column and row of $p_1 \times p_1$ blocks is distinguished. Identical shading means identical components. A blank means zero.

The evident symmetry between these vectors makes it obvious that pairwise they make the same angle which, together with (19), means that they span an $(n/p_1 - 1)$ -dimensional hypertetrahedron. It also follows that there is no further linear relationship between them, so if we discard one, say $|1; 0; n/p_1\rangle$, we will be left with

$$\mu(1) = n \left(\frac{1}{p_1} - \frac{1}{p_0} \right) \tag{20}$$

linearly independent basis vectors. These will not be normalized, nor orthogonal, however. It would be an easy task to construct an orthonormal set out of them, but it would destroy their symmetry. We find it slightly more convenient to work with a bi-orthogonal set. For the same reason we need not worry about normalization. It is an elementary exercise to show that the set

$$\frac{4p_1}{n^2(n - 2p_1)} \left(|1; 0; b\rangle - |1; 0; \frac{n}{p_1}\rangle \right) \quad b = 1, 2, \dots, \frac{n}{p_1} - 1 \tag{21}$$

is bi-orthogonal to the set $|1; 0; b\rangle$.

We now proceed, still within the $k = 1$ subfamily, to the next level of the Parisi hierarchy. The vectors $|k = 1; i = 1; b\rangle$ are shown in figure 2. Similarly to the previous case, orthogonality to the first family vector $|0; 1\rangle$ demands

$$\sum_{b=1}^{n/p_1} |1; 1; b\rangle = 0$$

which is satisfied if the vector components are chosen as

$$A = 1 \quad B = 1 - \frac{n}{p_1} \tag{22}$$

Then the vectors $|1; 1; b\rangle$ span a $\mu(1)$ -dimensional hypertetrahedron again, with the associated bi-orthogonal set

$$\frac{2p_1}{n^2(p_1 - p_2)} \left(|1; 1; b\rangle - |1; 1; \frac{n}{p_1}\rangle \right) \quad b = 1, 2, \dots, \frac{n}{p_1} - 1 \tag{23}$$

The construction proceeds along similar lines: filling in the i th level of the Parisi hierarchy we find the same $\mu(1)$ -dimensional tetrahedra with the same orthogonality conditions (22)

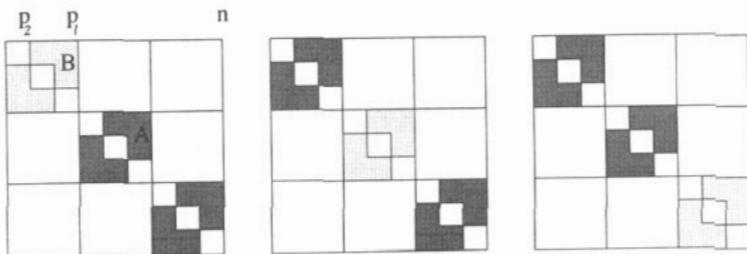


Figure 2. The basis vectors $|1; 1; b\rangle$.

applying for any $i \geq 1$ (only the case $i = 0$ (equation (18)) is different). This way we will have, altogether, $\mu(1)(R + 1)$ independent basis vectors

$$|1; i; b\rangle \quad i = 0, 1, \dots, R \quad \text{and} \quad b = 1, 2, \dots, \frac{n}{p_1} - 1$$

making up the $k = 1$ subfamily. They are each orthogonal to the first family, two of them belonging to different values of i are also orthogonal, but two such vectors with the same i and different b 's are not.

It can now be shown again that the subspace spanned by the $R + 1$ vectors $|1; i; b\rangle$, $i = 0, 1, \dots, R$, for b fixed is an invariant subspace of an arbitrary ultrametric matrix. Therefore the linear combination

$$|f\rangle = \sum_{i=0}^R f_1(i) |1; i; b\rangle$$

with appropriately chosen amplitudes $f_1(i)$, independent of b , will be an eigenvector. There will be $R + 1$ choices for these amplitudes, yielding $R + 1$ eigenvalues $\lambda_1(m)$, $m = 0, 1, \dots, R$, in the $k = 1$ subfamily. Each of these will be $\mu(1)$ -fold degenerate, according to the free choice of b .

We now turn to the $k = 2$ subfamily. Some of the ($k = 2$)-type vectors with the $i = 0$ Parisi level filled are shown in figure 3. These vectors will be labelled as $|2; 0; a, b\rangle$, where $a = 1, 2, \dots, n/p_1$ shows which column of $p_1 \times p_1$ -sized blocks has non-zero elements and $b = 1, 2, \dots, p_1/p_2$ shows which column of $p_2 \times p_2$ -sized blocks is distinguished inside column a . The sum $\sum_{b=1}^{p_1/p_2} |2; 0; a, b\rangle$ must vanish again for any fixed a , otherwise it would be a linear combination of $k = 0$ and $k = 1$ subfamily type vectors. This orthogonality condition demands that we choose

$$A = 1 \quad B = 1 - \frac{p_1}{p_2} \tag{24}$$

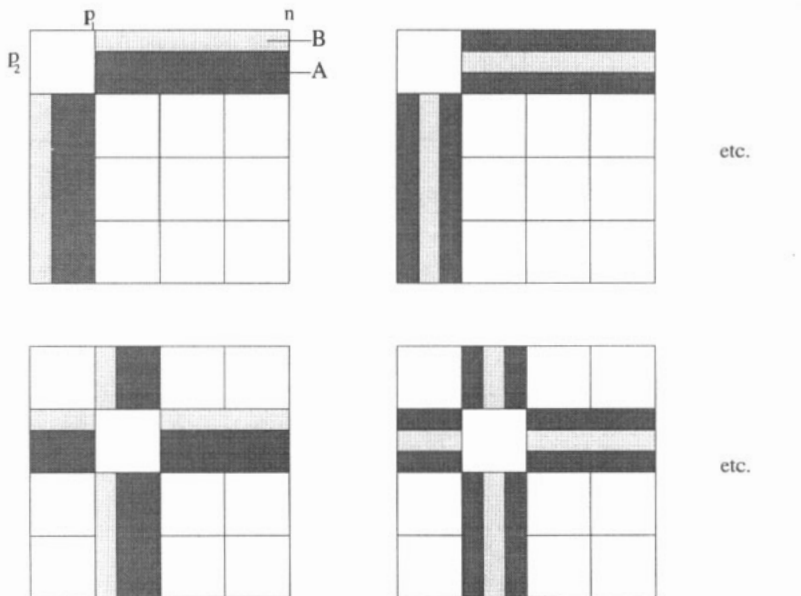


Figure 3. Some vectors of the $k = 2$ subfamily ($i = 0$)

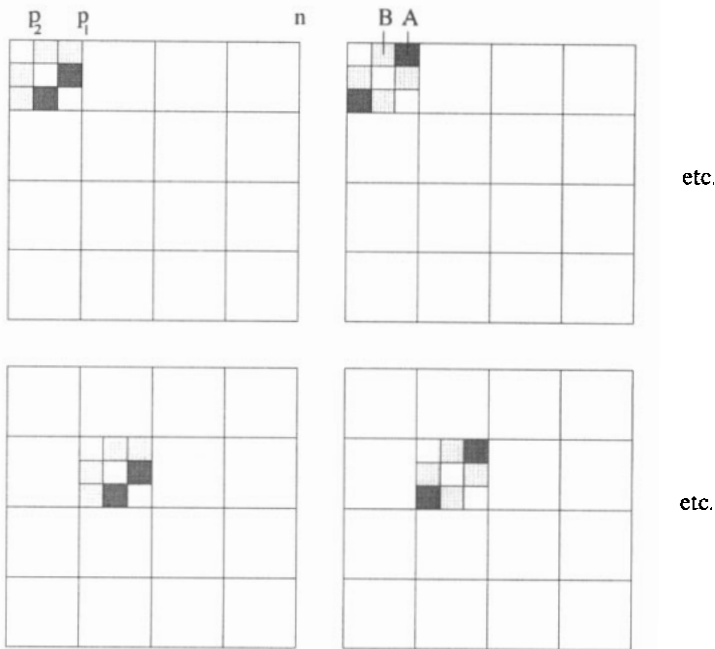


Figure 4. Some vectors of the $k = 2$ subfamily ($i = 1$).

leaving $p_1/p_2 - 1$ independent vectors (spanning a tetrahedron again) for any fixed a . It is easy to see that with this choice the vectors $|2; 0; a, b\rangle$ will not only be orthogonal to each of the previous families (with $k = 0, 1$) but they will also be orthogonal to the vectors $|2; 0; a', b'\rangle$ with $a \neq a'$ and any b' .

Some $k = 2, i = 1$ vectors are shown in figure 4. The orthogonality conditions now read

$$A = 1 \quad B = \frac{1}{2} \left(2 - \frac{p_1}{p_2} \right).$$

We can go on to build $|2; i; a, b\rangle, i = 0, 1, \dots, R$ in a similar manner. The subspace of these vectors for fixed a and b will be closed under the action of an ultrametric matrix M . Thus

$$\sum_{i=0}^R f_2(i) |2; i; a, b\rangle$$

will be an eigenvector with $R + 1$ choices for the amplitudes and with eigenvalues $\lambda_2(m), m = 0, 1, \dots, R$. We have seen that for a given a we have $p_1/p_2 - 1$ linearly independent choices for the basis vectors, while for different a 's they are already orthogonal. That means we have $(n/p_1)(p_1/p_2 - 1)$, i.e.

$$\mu(2) = n \left(\frac{1}{p_2} - \frac{1}{p_1} \right)$$

independent basis vectors for any i . The total dimension of the $k = 2$ subfamily is thus $\mu(2)(R + 1)$, and the multiplicity of the $k = 2$ eigenvalues is $\mu(2)$.

The generalization is now obvious. In the k th subfamily ($k = 1, 2, \dots, R + 1$) we have vectors labelled by four indices: $|k; i; a, b\rangle$. This vector has non-zero elements only on the i th level of the Parisi hierarchy, and there only inside one column and row of blocks of size $p_{k-1} \times p_{k-1}$. There are n/p_{k-1} such columns, and the label $a = 1, 2, \dots, n/p_{k-1}$ shows which is the one in question. The vector components inside these blocks are all 1's, except inside a distinguished column and row of blocks of size $p_k \times p_k$ where they are†

$$B = \frac{1}{2} \left(2 - \frac{p_{k-1}}{p_k} \right) \quad \text{if } i = k - 1$$

$$B = 1 - \frac{p_{k-1}}{p_k} \quad \text{otherwise.} \quad (25)$$

The distinguished columns of $p_k \times p_k$ blocks are labelled by the last index, $b = 1, 2, \dots, p_{k-1}/p_k$.

With the choice (25), the vectors $|k; i; a, b\rangle$ are orthogonal to all the previous subfamilies with $k-1, k-2, \dots$ etc, down to the first family. Within the k th subfamily vectors belonging to different i 's and a 's are also orthogonal, while those with a fixed k, i, a and different b 's make a $(p_{k-1}/p_k - 1)$ -dimensional hypertetrahedron. The number of linearly independent vectors for a given k and i will then be $(n/p_{k-1})(p_{k-1}/p_k - 1)$, i.e.

$$\mu(k) = n \left(\frac{1}{p_k} - \frac{1}{p_{k-1}} \right). \quad (26)$$

The bi-orthogonal set associated with the tetrahedral groups of vectors belonging to a given triplet k, i, a is

$$|k; \widetilde{i}; a, b\rangle = \frac{p_k}{p_{k-1}^2} \frac{1}{g_i^{(k)}} \left(|k; i; a, b\rangle - |k; i; a, \frac{p_{k-1}}{p_k} \rangle \right) \quad (27)$$

where the weight $g_i^{(k)}$ is defined as

$$g_i^{(k)} = \begin{cases} p_i - p_{i+1} & i < k - 1 \\ \frac{1}{4}(p_{k-1} - 2p_k) & i = k - 1 \\ \frac{1}{2}(p_i - p_{i+1}) & i > k - 1. \end{cases} \quad (28)$$

For fixed k, a, b the $R + 1$ vectors $|k; i; a, b\rangle, i = 0, 1, \dots, R$, form an invariant subspace of any ultrametric matrix, so we will have eigenvectors of the form

$$\sum_{i=0}^R f_k(i) |k; i; a, b\rangle \quad (29)$$

with amplitudes $f_k(i)$ independent of a and b . The corresponding eigenvalue equations will give $R + 1$ possible values for $f_k(i)$, and the eigenvalues $\lambda_m(k), m = 0, 1, \dots, R$, will be $\mu(k)$ -fold degenerate. Sometimes the second family is also called the anomalous or A family.

† We take the opportunity here to correct a misprint in equation (4) of [13]. The B for $i = k - 1$ was given as $B = \frac{1}{2}(1 - p_{k-1}/p_k)$ there, instead of the correct expression in (25).

So far, from the $\frac{1}{2}n(n - 1)$ -dimensional replica space we have split off the $(R + 1)$ -dimensional invariant subspace of the first family, the $(R + 1)$ -dimensional subspaces, $\mu(1)$ in number, of the $k = 1$ subfamily, etc, up to $k = R + 1$; that is, we have decomposed our linear space into

$$\sum_{k=0}^{R+1} \mu(k) = n$$

$(R + 1)$ -dimensional invariant subspaces plus the vast space, of dimension $\frac{1}{2}n(n - 1) - n(R + 1)$, orthogonal to the first and second families.

The third family.

The third family, often called the replicon or R family, comprises everything remaining after splitting off the first two families. It is a most remarkable fact, and a direct consequence of the stringent conditions ultrametricity imposes upon a matrix, that the third family can be decomposed into invariant subspaces of dimension 1, i.e. directly into eigenvectors. This also means that the third family eigenvalues that, for large n , represent the overwhelming majority of all the eigenvalues can be obtained in closed form, in terms of the matrix elements, for any ultrametric matrix.

The third family eigenvectors were given in a concise form in [13]. We provide a little more detail here which will become important when we invert the matrix M .

The third family consists of several subfamilies labelled by three integers

$$r = 0, 1, \dots, R \quad k, l = r + 1, r + 2, \dots, R + 1.$$

There will be several degenerate vectors in each subfamily. They will be labelled by three, five, or seven more indices, as the need arises. A common property of all third family vectors is that they have non-zero components only inside one single diagonal block of size $p_r \times p_r$, which also gives the significance of the label r above. The labels k and l specify further structural details that are best displayed on a series of figures again. In the following we shall exhibit only that $p_r \times p_r$ -sized block over which the vector components are not all zero.

The $r, k = r + 1, l = r + 1$ subfamily.

The structure of the nonvanishing block is shown in figure 5. These vectors take three further labels to specify them: $|r; r + 1, r + 1; a, b, c\rangle$. The index $a = 1, 2, \dots, n/p_r$ shows which of the n/p_r diagonal $p_r \times p_r$ blocks has non-vanishing elements. Inside this block all the components belonging to the diagonal $p_{r+1} \times p_{r+1}$ blocks vanish again. Of the off-diagonal $p_{r+1} \times p_{r+1}$ blocks, those belonging to two columns and rows are distinguished and a further distinction is made between the blocks at the crossing of a distinguished column and row and the rest. The indices $b, c = 1, 2, \dots, p_r/p_{r+1}, b \neq c$, label the two distinguished columns. In all, we then have three different vector components in this subfamily, as shown in the figure.

Orthogonality to the previous families requires that

$$\sum_{\substack{b=1 \\ b \neq c}}^{p_r/p_{r+1}} |r; r + 1, r + 1; a, b, c\rangle = 0 \quad \sum_{\substack{c=1 \\ c \neq b}}^{p_r/p_{r+1}} |r; r + 1, r + 1; a, b, c\rangle = 0. \tag{31}$$

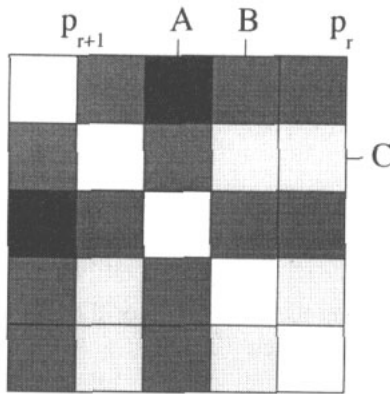


Figure 5. The third family vector $|r; r + 1, r + 1; a, b, c\rangle$. Identical shading means identical components; a blank means zero. $b = 1, c = 3$.

It follows from (31) that the sum of vector components in each row should vanish, giving us two equations for the three numbers A, B, C , the third being determined by normalization. They work out to be

$$\begin{aligned}
 B &= -\frac{p_{r+1}}{p_r - 2p_{r+1}} A \\
 C &= \frac{2p_{r+1}^2}{(p_r - 2p_{r+1})(p_r - 3p_{r+1})} A \\
 A^2 &= \frac{p_r - 3p_{r+1}}{p_{r+1}^2(p_r - p_{r+1})}.
 \end{aligned}
 \tag{32}$$

With this we have determined the eigenvectors with $r, r + 1, r + 1$ completely. The corresponding eigenvalues will be written up in the next section. For a given position of the $p_r \times p_r$ block, i.e. for a given a , the orthogonality conditions (31) leave $\frac{1}{2}(p_r/p_{r+1})(p_r/p_{r+1} - 3)$ vectors linearly independent. This number, multiplied by n/p_r , the number of choices for a , gives the multiplicity of this class:

$$\mu(r; r + 1, r + 1) = \frac{1}{2}n \frac{p_r - 3p_{r+1}}{p_{r+1}^2} \quad r = 0, 1, \dots, R.
 \tag{33}$$

Eigenvectors with $r, k > r + 1, l = r + 1$.

An example is shown in figure 6. Such a vector is constructed as follows. One chooses a diagonal block of size $p_r \times p_r$, labelled by $a = 1, 2, \dots, n/p_r$, as before. Inside this block one chooses two columns and rows of blocks of size $p_{r+1} \times p_{r+1}$, say the b th and the c th, such that $c > b$ ($b = 1, c = 2$ in figure 6). Inside the blocks in the b th column and row, one now chooses a strip of blocks of size $p_{k-1} \times p_{k-1}$, say the d th, as shown in the figure. All the vector components outside this strip will be zero. Inside the strip one chooses a strip, the e th, of blocks of size $p_k \times p_k$. Finally the vector components A, B, C, D are arranged, as shown, according to whether they belong to the strip of width p_k or whether they are outside, and also whether they belong to the c th blocks or not.

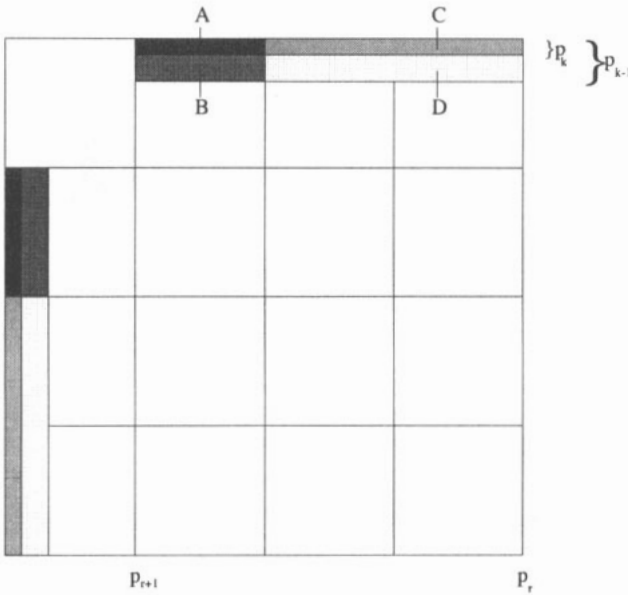


Figure 6. An eigenvector of the class $|r; k \geq r + 2, l = r + 1; a, b, c, d, e)$.

Orthogonality to previous families again requires, as throughout the third family, that the sum of components in each row vanishes. This immediately gives

$$\begin{aligned}
 B &= -\frac{p_k}{p_{k-1} - p_k} A \\
 C &= -\frac{p_{r+1}}{p_r - 2p_{r+1}} A \\
 D &= \frac{p_k p_{r+1}}{(p_{k-1} - p_k)(p_r - 2p_{r+1})} A \\
 A^2 &= \frac{p_r - 2p_{r+1}}{p_{r+1}(p_r - p_{r+1})} \left(\frac{1}{p_k} - \frac{1}{p_{k-1}} \right)
 \end{aligned}
 \tag{34}$$

for the components of a normalized vector of this class.

Considering the various choices for the parameters a, b, c, d, e , one finds that the total number of linearly independent vectors of this kind is

$$\mu(r; k, r + 1) = \frac{1}{2} n \frac{p_r - 2p_{r+1}}{p_{r+1}} \left(\frac{1}{p_k} - \frac{1}{p_{k-1}} \right) \quad k = r + 2, r + 3, \dots, R + 1. \tag{35}$$

The subfamily $r; k = r + 1, l > r + 1$ is obtained by the same construction but with $c < b$; this subfamily will be similar to the $r; k > r + 1, l = r + 1$ subfamily in every respect, but it will be orthogonal to it.

Eigenvectors with $r, k > r + 1, l > r + 1$.

The construction of these vectors is shown in figure 7. It begins again by choosing one diagonal block of size $p_r \times p_r$, labelled by a . This can be done in n/p_r different ways. Next, inside this block one chooses two symmetrically positioned off-diagonal blocks of

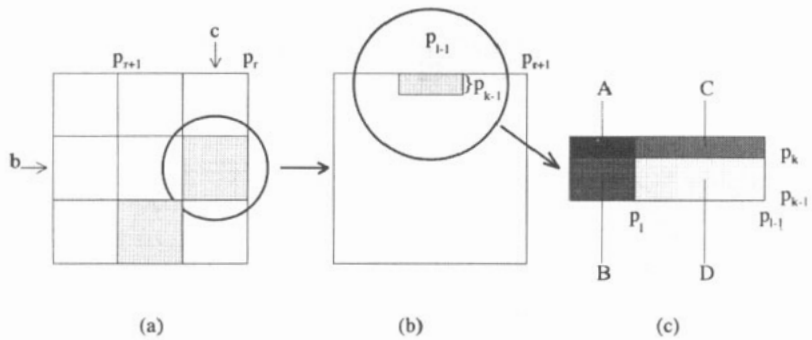


Figure 7. The construction of an eigenvector with $r, k > r + 1, l > r + 1$. The structure of the vector is shown in the three stages of finer and finer resolution. The block with indices c, b in (a) is obtained by reflection to the diagonal from block b, c .

size $p_{r+1} \times p_{r+1}$. This takes two indices: $b, c = 1, 2, \dots, p_r/p_{r+1}$, and the number of independent choices is $\frac{1}{2}(p_r/p_{r+1})(p_r/p_{r+1} - 1)$, because of the symmetry of the matrix representing the eigenvector. Now the off-diagonal $p_{r+1} \times p_{r+1}$ block above the diagonal is cut into rectangles of horizontal size p_{l-1} and vertical size p_{k-1} and the one below the diagonal is cut similarly with the horizontal and vertical dimensions exchanged.

One of these rectangles is chosen, which again takes two labels: $d = 1, 2, \dots, p_{r+1}/p_{l-1}$, and $e = 1, 2, \dots, p_{r+1}/p_{k-1}$, and can be done in $p_{r+1}^2/(p_{l-1}p_{k-1})$ ways. **We have non-zero components only inside these rectangles.** Their structure is shown in figure 7(c), and can evidently be characterized by two further indices $f = 1, 2, \dots, p_{l-1}/p_l$, and $g = 1, 2, \dots, p_{k-1}/p_k$. These vectors are orthogonal to each other in all the indices except for f and g . For fixed f the set with different g 's forms the usual tetrahedron again, and the same is true for fixed g in the f 's, so we are left with $(p_{k-1}/p_k - 1) \times (p_{l-1}/p_l - 1)$ independent choices for f and g . All these taken together give a multiplicity

$$\mu(r; k, l) = \frac{1}{2}n(p_r - p_{r+1}) \left(\frac{1}{p_k} - \frac{1}{p_{k-1}} \right) \left(\frac{1}{p_l} - \frac{1}{p_{l-1}} \right) \quad (36)$$

$$(k, l = r + 2, r + 3, \dots, R + 1)$$

while the usual orthogonality conditions (the sum of vector components in each row and each column vanishes) give us the following values for the components of a normalized vector of the $r, k > r + 1, l > r + 1$ type:

$$B = -\frac{p_k}{p_{k-1} - p_k} A$$

$$C = -\frac{p_l}{p_{l-1} - p_l} A$$

$$D = \frac{p_k p_l}{(p_{k-1} - p_k)(p_{l-1} - p_l)} A \quad (37)$$

$$A^2 = \left(\frac{1}{p_k} - \frac{1}{p_{k-1}} \right) \left(\frac{1}{p_l} - \frac{1}{p_{l-1}} \right).$$

With this we have given a full description of the third family eigenvectors. To show that these forms are indeed reproduced under the action of an ultrametric matrix takes, of course, a lot of algebra. It is impossible for us to go into details on this, but we think a little experimentation in a simple special case like $R = 1$ will convince the reader that the proof is quite straightforward though certainly not very short. As we have seen, there is a high degree of degeneracy within each subfamily $(r; k, l)$. These subfamilies are all orthogonal to each other (and also to the first and second families, of course), and some of the degenerate vectors within a given $(r; k, l)$ subfamily are also orthogonal, but some form the by now familiar tetrahedral sets in more than one index. It would not be difficult to orthogonalize these vectors, or, alternatively, to construct bi-orthogonal sets to them. We refrain from doing both: the loss in symmetry would be considerable and the gain virtually nothing. The only occasion when we might need a properly orthonormalized set is when later we construct the ‘replicon’ components of the inverse of M from the spectral resolution. It will turn out, however, that the orthogonalization can be circumvented even there and the ultrametric symmetries of M (and of its inverse) will allow us to deduce the full contribution of the whole $(r; k, l)$ subfamily to the inverse from the knowledge of a single vector belonging to that subfamily. This vector will be called the *representative vector* of the subfamily.

We can choose any of the $\mu(r; k, l)$ degenerate vectors to be the representative vector. Suppose we have made our choice. Some of the vectors in the $(r; k, l)$ subfamily will be orthogonal to the selected vector from the beginning. These will, as a rule, have zero components where the selected vector has non-zero ones; in particular, they will have zeros where the components we called A ’s in the description of the third family vectors, i.e. those in the darkest shaded areas in the figures, are to be found in the selected vector. Now consider the vectors which are not orthogonal to the selected one. In order to orthogonalize them to the representative vector we form some linear combinations. Our key observation is now that, due to the defining properties of the third family vectors (namely, that the sums of vector components in each row are zero), any linear combination that is orthogonal to the representative vector will have zero components where the representative vector has its components A . The proof of this will also be left as an exercise to the reader.

The last issue to be settled in this section is the total multiplicity of our basis vectors. In the first two families we found $n(R + 1)$ independent basis vectors. In the subfamilies $(r; k, l)$, for fixed r , there are a total of

$$\sum_{k,l=r+1}^{R+1} \mu(r; k, l) = \frac{1}{2}n(p_r - p_{r+1} - 2) \tag{38}$$

independent vectors. This summed over r gives the total number of basis vectors in the third family:

$$\sum_{r=0}^R \frac{1}{2}n(p_r - p_{r+1} - 2) = \frac{1}{2}n(n - 2R - 3). \tag{39}$$

Added to $n(R + 1)$ this gives $\frac{1}{2}n(n - 1)$, the dimension of replica space. Our set of basis vectors is therefore complete.

4. The block-diagonal form of ultrametric matrices

Having constructed a complete set of basis vectors we can build a matrix S with columns made of these vectors and transform M to this new basis by

$$\tilde{M} = S^{-1}MS. \tag{40}$$

S is not an orthogonal transformation (because the basis vectors are not all orthogonal), so \tilde{M} will not be symmetric. The rows of the inverse S^{-1} in the first family sector will be made of the bra vectors corresponding to the first-family-like basis vectors $|0; i\rangle$; in the second family sector they will be the bi-orthogonal vectors given in (27). In the third family sector we do not really need to construct the matrix S^{-1} at all, since there the basis vectors are the eigenvectors themselves. Since the various families and subfamilies were constructed in such a way that they are invariant subspaces of M , the transformed matrix \tilde{M} will have a block-diagonal form: along the diagonal we will have a string of $n(R + 1) \times (R + 1)$ matrices, the first corresponding to the first family, the next $\mu(1)$ identical matrices corresponding to the $k = 1$ subfamily in the second family, etc, through $\mu(k)$ identical blocks for the k th subfamily up to $k = R + 1$. This string of matrices will be followed by the string of the third family eigenvalues coming in groups of $\mu(r; k, l)$ identical numbers corresponding to the subfamilies $(r; k, l)$.

The third family or replicon eigenvalues are obtained as a byproduct of checking that the third family vectors given in the previous section are eigenvectors indeed. In the $(r; k, l)$ subfamily one obtains the closed expression

$$\lambda(r; k, l) = \sum_{s=k}^{R+1} p_s \sum_{t=l}^{R+1} p_t (M_{t,s}^{r,r} - M_{t-1,s}^{r,r} - M_{t,s-1}^{r,r} + M_{t-1,s-1}^{r,r}) \tag{41}$$

$r = 0, 1, \dots, R, \quad k, l = r + 1, r + 2, \dots, R + 1$

giving the replicon eigenvalues directly in terms of the matrix elements and of the p 's characterizing the structure of M . Equation (41) has been written up already in [13].

The $(R + 1) \times (R + 1)$ diagonal blocks of \tilde{M} will be labelled by $k = 0, 1, \dots, R + 1$ as $M^{(k)}$, $k = 0$ corresponding to the first family, $k > 0$ to the subfamilies in the second family. The matrix elements of $M^{(0)}$ can be obtained by sandwiching M between two first family vectors

$$M_{r,s}^{(0)} = \langle 0; r | M | 0; s \rangle \tag{42}$$

those of $M^{(k)}$ by sandwiching M between a second family vector and one from the bi-orthogonal set given in (27):

$$M_{r,s}^{(k)} = \langle k; r; a, b | M | k; s; a, b \rangle \tag{43}$$

and can really be obtained again as byproducts when verifying the invariance of the various subfamilies under the action of the matrix M .

We can therefore simply state the result

$$M_{r,s}^{(k)} = \Lambda(k, r) \delta_{r,s}^{K_r} + g_s^{(k)} \frac{(\Delta_r^{(k)} + \frac{1}{2} p_{k-1} \delta_{r,k-1}^{K_r})(\Delta_s^{(k)} + \frac{1}{2} p_{k-1} \delta_{s,k-1}^{K_r})}{\delta_r \delta_s} K_k(r, s). \tag{44}$$

Of the symbols appearing here δ_r and $g_s^{(k)}$ have already been defined in (16) and (28), respectively. $\delta_{r,s}^{K_r}$ is the Kronecker symbol, while $\Delta_r^{(k)}$ is

$$\Delta_r^{(k)} = \begin{cases} \frac{1}{2} \delta_r & r < k - 1 \\ \frac{1}{2} (\delta_{k-1} - p_k) & r = k - 1 \\ \delta_r & r > k - 1. \end{cases} \tag{45}$$

The diagonal part $\Lambda(k, r)$ is related to the third family eigenvalues:

$$\Lambda(k, r) = \begin{cases} \lambda(r; r + 1, k) & k > r + 1 \\ \lambda(r; r + 1, r + 1) & k \leq r + 1. \end{cases} \quad (46)$$

Now we define the kernel $K_k(r, s)$ appearing in (44):

$$\begin{aligned} \frac{1}{4} K_k(r, s) = & \frac{1}{4} \sum_{j=k}^r p_j (M_{j,j}^{r,s} - M_{j-1,j-1}^{r,s}) + \frac{1}{2} \sum_{j=r+1}^s p_j (M_{r,j}^{r,s} - M_{r,j-1}^{r,s}) \\ & + \sum_{j=s+1}^{R+1} p_j (M_{r,j}^{r,s} - M_{r,j-1}^{r,s}) \quad k - 1 \leq r \leq s \end{aligned} \quad (47)$$

$$\begin{aligned} \frac{1}{4} K_k(s, r) = & \frac{1}{4} \sum_{j=k}^r p_j (M_{j,j}^{s,r} - M_{j-1,j-1}^{s,r}) + \frac{1}{2} \sum_{j=r+1}^s p_j (M_{j,j}^{s,r} - M_{j-1,j-1}^{s,r}) \\ & + \sum_{j=s+1}^{R+1} p_j (M_{s,j}^{s,r} - M_{s,j-1}^{s,r}) \quad k - 1 \leq r \leq s \end{aligned} \quad (48)$$

$$\frac{1}{4} K_k(r, s) = \frac{1}{2} \sum_{j=k}^s p_j (M_{r,j}^{r,s} - M_{r,j-1}^{r,s}) + \sum_{j=s+1}^{R+1} p_j (M_{r,j}^{r,s} - M_{r,j-1}^{r,s}) \quad r \leq k - 1 \leq s \quad (49)$$

$$\frac{1}{4} K_k(s, r) = \frac{1}{2} \sum_{j=k}^s p_j (M_{j,j}^{s,r} - M_{j-1,j-1}^{s,r}) + \sum_{j=s+1}^{R+1} p_j (M_{s,j}^{s,r} - M_{s,j-1}^{s,r}) \quad r \leq k - 1 \leq s \quad (50)$$

$$\frac{1}{4} K_k(r, s) = \sum_{j=k}^{R+1} p_j (M_{r,j}^{r,s} - M_{r,j-1}^{r,s}) \quad r \leq s \leq k - 1 \quad (51)$$

$$\frac{1}{4} K_k(s, r) = \sum_{j=k}^{R+1} p_j (M_{s,j}^{s,r} - M_{s,j-1}^{s,r}) \quad r \leq s \leq k - 1. \quad (52)$$

(Possible empty sums here and in the following are understood to be zero. For $k = 0$, terms with $M_{-1,-1}^{r,s}$ may occur in the above formulae. They are, by definition, zero too.)

With this the matrix elements $M_{r,s}^{(k)}$ in the new representation have been expressed in terms of the matrix elements in the original representation. The problem of finding the eigenvalues of M in the first two families has thus been broken down into the problem of finding the spectrum of each of the $M^{(k)}$'s. As we have already noted, due to the non-orthogonality of the transformation S , the $M^{(k)}$'s are not symmetric. Using the symmetries between the various components of M as described in section 2, one can show, however, that, although we have given the expressions for both $K_k(r, s)$ and $K_k(s, r)$ in the various cases for completeness, the kernel K_k is, in fact, symmetric. Therefore, the asymmetry of $M^{(k)}$ is carried solely by the factor $g_k^{(s)}$ in (44), and we can, if we wish, reduce the eigenvalue problem of $M^{(k)}$ to that of the manifestly symmetric matrix

$$g_r^{(k)1/2} \frac{M_{r,s}^{(k)}}{g_s^{(k)}} g_s^{(k)1/2}.$$

Let us now spell out the eigenvalue equation of $M^{(k)}$:

$$\sum_{n=0}^R M_{r,n}^{(k)} f_k(s) = \lambda(k) f_k(r)$$

reads in the two cases $r < k - 1$ and $r \geq k - 1$, as, respectively,

$$\begin{aligned} \Lambda(k, r) f_k(r) + \frac{1}{4} \sum_{\substack{s=0 \\ s \neq k-1}}^R K_k(r, s) f_k(s) \delta_s + \frac{1}{8} (\delta_{k-1} - p_k) K_k(r, k-1) f_k(k-1) \\ = \lambda(k) f_k(r) \quad r < k - 1 \end{aligned} \tag{53}$$

$$\begin{aligned} \Lambda(k, r) f_k(r) + \frac{1}{2} \sum_{\substack{s=0 \\ s \neq k-1}}^R K_k(r, s) f_k(s) \delta_s + \frac{1}{4} (\delta_{k-1} - p_k) K_k(r, k-1) f_k(k-1) \\ = \lambda(k) f_k(r) \quad r \geq k - 1. \end{aligned}$$

This is a set of $R + 1$ homogeneous linear equations for any given $k = 0, 1, \dots, R + 1$. (In the first family, i.e. for $k = 0$, only the equation with $r > k - 1$ applies, with the third term on the LHS discarded.) The solutions for the f_k 's give the amplitudes mentioned in the preceding section, while the $\lambda(k)$'s are the corresponding eigenvalues.

It is evident from the definition that the space of ultrametric matrices (belonging to the same series p_0, p_1, \dots, p_R) is closed under addition. That it is also closed under multiplication is easiest to see from the existence of the common similarity transformation S that brings any two such matrices to block-diagonal form simultaneously.

Suppose we are given two ultrametric matrices M and M' , with the associated kernels K_k and K'_k and replicon eigenvalues λ^R and $\lambda^{R'}$ from which we have the corresponding Λ 's as given in (46). Then in the block-diagonal representation of the product MM' we will find for the diagonal blocks in the LA sector:

$$\begin{aligned} \sum_{t=0}^R M_{r,t}^{(k)} M'_{t,s}{}^{(k)'} = g_s^{(k)} \frac{(\Delta_r^{(k)} + \frac{1}{2} p_{k-1} \delta_{r,k-1}^{Kr}) (\Delta_s^{(k)} + \frac{1}{2} p_{k-1} \delta_{s,k-1}^{Kr})}{\delta_r \delta_s} \\ \times \left\{ \frac{1}{2} \sum_{t=0}^R \Delta_t^{(k)} K_k(r, t) K'_k(t, s) + \Lambda(k, r) K'_k(r, s) + \Lambda'(k, s) K_k(r, s) \right\} \\ + \delta_{r,s}^{Kr} \Lambda(k, r) \Lambda'(k, r) \end{aligned} \tag{54}$$

where we have used that by (16), (28) and (45)

$$g_t^{(k)} \frac{(\Delta_t^{(k)} + \frac{1}{2} p_{k-1} \delta_{t,k-1}^{Kr})^2}{\delta_t^2} \equiv \frac{1}{2} \Delta_t^{(k)} \tag{55}$$

whereas in the R sector we will evidently find the product of the replicon eigenvalues:

$$\lambda(r; k, l) \lambda'(r; k, l). \tag{56}$$

In particular, if M' is the inverse of M , we have $\sum_{t=0}^R M_{r,t}^{(k)} M'_{t,s}{}^{(k)} = \delta_{r,s}^{Kr}$ and $\Lambda(k, r) \Lambda'(k, r) = 1$, so from (54) we get

$$\frac{1}{2} \sum_{t=0}^R \Delta_t^{(k)} K_k(r, t) K'_k(t, s) + \Lambda(k, r) K'_k(r, s) + \Lambda'(k, s) K_k(r, s) = 0. \tag{57}$$

For a given M , i.e. for a given K_k and $\Lambda(k, r)$, this is an equation for the kernel K'_k of the inverse matrix. In the general case equation (57) is still a matrix equation which is typically difficult to solve, though (57) certainly has the merit of reducing the problem of the inversion of a $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ -dimensional matrix to that of inverting $R+2$ much smaller matrices (corresponding $k = 0, 1, \dots, R+1$ in (57)) of size $(R+1) \times (R+1)$. In some important special cases, however, further progress can be made and both the solution of the eigenvalue equations (53) and the inversion of the matrix M can be carried through to the end [12].

Coming back to (57) let us assume now that we have somehow succeeded in solving it for K'_k . With this the inversion of M is, however, not yet completed, because normally we need the inverse in the original Cartesian coordinate system, i.e. we need the components of M' given the kernel K'_k . This means we have to invert the formulae (47)–(52), or, to put it even more simply, we have to turn (40) around like

$$M = \tilde{S} \tilde{M} S^{-1}. \tag{58}$$

In the LA sector this is a standard operation: we have all the basis vectors and their bi-orthogonal counterparts so that we explicitly know the corresponding blocks in S and S^{-1} . This is not the case in the R sector where we have neither orthogonalized nor bi-orthogonalized our basis vectors. In the direct transformation, from M to \tilde{M} , this did not cause a problem, because, the third family basis vectors being eigenvectors, we knew in advance that the corresponding ‘blocks’ of \tilde{M} would be the eigenvalues themselves. In the inverse transformation, from \tilde{M} to M , however, we would definitely need the missing blocks in S^{-1} in order to determine the contribution of the replicon family to the various components of M . It is at this point that the concept of the representative vector introduced in the previous section becomes important. We do not think we should dwell upon how the blocks of the three matrices in (58) have to be multiplied in the sector where they are known. We have to explain, however, how the replicon contribution to (58) can be obtained from the representative vectors without actually knowing the corresponding block in S^{-1} . The appendix is devoted to this problem.

In what follows we will state our results for the nine different types of matrix elements of M discussed in section 2 in terms of the matrix elements of the block-diagonal form \tilde{M} . In each case we shall give the result in two different forms: first as a sum of two terms, one coming from the LA sector, the other from the replicon, and second, in a form where some most remarkable cancellations between these two have been effected. In the discrete case, where n , R , and all the p_k 's are integers, these cancellations may seem coincidental. We note, however, that in the continuous limit they acquire a fundamental importance [13].

In order to display these cancellations we partition the third family multiplicities as follows:

$$\mu(r; k, l) \equiv \mu_{\text{reg}}(r; k, l) + \mu_{\text{sing}}(r; k, l) \tag{59}$$

where

$$\mu_{\text{reg}}(r; r+1, r+1) = \frac{1}{2}n \frac{p_r}{p_{r+1}} \left(\frac{1}{p_{r+1}} - \frac{1}{p_r} \right) \tag{60}$$

$$\mu_{\text{sing}}(r; r+1, r+1) = - \sum_{k=0}^{r+1} \mu(k) \tag{61}$$

$$\mu_{\text{reg}}(r; r+1, k) = \mu_{\text{reg}}(r; k, r+1) = \frac{1}{2}n \frac{p_r - p_{r+1}}{p_{r+1}} \left(\frac{1}{p_k} - \frac{1}{p_{k-1}} \right) \quad k > r+1 \tag{62}$$

$$\mu_{\text{sing}}(r; r+1, k) = \mu_{\text{sing}}(r; k, r+1) = -\frac{1}{2}\mu(k) \quad k > r+1 \tag{63}$$

and $\mu_{\text{reg}}(r; k, l)$ for $k, l > r + 1$ is the full $\mu(r; k, l)$ itself, as given in (36), so that

$$\mu_{\text{sing}}(r; k, l) = 0 \quad k, l > r + 1. \tag{64}$$

In equations (61) and (63) $\mu(k)$ is the second family multiplicity. In the discrete case the subscripts ‘regular’ and ‘singular’ have no particular significance; in the continuous limit, however, μ_{sing} will be associated with terms that become meaningless but disappear from the theory due to the cancellations mentioned above.

We now list the results:

$$M_{R+1, R+1}^{r, r} = \sum_{k=0}^{R+1} \frac{2\mu(k)}{n\delta_r} M_{r, r}^{(k)} + \sum_{k=r+1}^{R+1} \sum_{l=r+1}^{R+1} \frac{2\mu(r; k, l)}{n\delta_r} \lambda(r; k, l) \tag{65a}$$

where the first term is the LA and the second the R contribution as announced. Substituting (44) for $M^{(k)}$ and splitting μ as $\mu = \mu_{\text{reg}} + \mu_{\text{sing}}$ we see that the $\Lambda(k, r)$ term coming from the LA cancels the μ_{sing} contributions from the R family exactly. So we have the alternative form:

$$M_{R+1, R+1}^{r, r} = \sum_{k=0}^{R+1} \frac{\mu(k)}{n\delta_r} \Delta_r^{(k)} K_k(r, r) + \sum_{k=r+1}^{R+1} \sum_{l=r+1}^{R+1} \frac{2\mu_{\text{reg}}(r; k, l)}{n\delta_r} \lambda(r; k, l). \tag{65b}$$

Similarly,

$$M_{R+1, t}^{r, r} = \sum_{k=0}^{R+1} \frac{2\mu(k)}{n\delta_r} \frac{\Delta_r^{(k)}}{\delta_t} M_{r, r}^{(k)} + \sum_{k=r+1}^{R+1} \sum_{l=r+1}^{R+1} \frac{2\mu(r; k, l)}{n\delta_r} \lambda(r; k, l) \left(\frac{\Delta_r^{(k)}}{\delta_t} + \frac{\Delta_t^{(l)}}{\delta_t} - 1 \right) \tag{66a}$$

$$= \sum_{k=0}^{R+1} \frac{\mu(k)}{n} \frac{\Delta_r^{(k)} \Delta_t^{(k)}}{\delta_r \delta_t} K_k(r, r) + \sum_{k=r+1}^{R+1} \sum_{l=r+1}^{R+1} \frac{2\mu_{\text{reg}}(r; k, l)}{n\delta_r} \lambda(r; k, l) \left(\frac{\Delta_r^{(k)}}{\delta_t} + \frac{\Delta_t^{(l)}}{\delta_t} - 1 \right) \quad t > r. \tag{66b}$$

In the special case $t = r$ of the above, things work out slightly differently:

$$M_{R+1, r}^{r, r} = \sum_{k=0}^{R+1} \frac{2\mu(k)}{n\delta_r} \frac{\Delta_r^{(k)} - \frac{1}{4} p_{k-1} \delta_{r, k-1}^{\text{Kr}}}{\delta_r - \frac{1}{2} p_{k-1} \delta_{r, k-1}^{\text{Kr}}} M_{r, r}^{(k)} - \frac{p_{r+1}}{p_r - 2p_{r+1}} \left\{ \frac{2\mu(r; r+1, r+1)}{n\delta_r} \lambda(r; r+1, r+1) + \sum_{k=r+2}^{R+1} \frac{2\mu(r; r+1, k)}{n\delta_r} \lambda(r; r+1, k) \right\} \tag{67a}$$

$$= \sum_{k=0}^{R+1} \frac{\mu(k)}{n\delta_r} \frac{\Delta_r^{(k)} - \frac{1}{4} p_{k-1} \delta_{r, k-1}^{\text{Kr}}}{\delta_r - \frac{1}{2} p_{k-1} \delta_{r, k-1}^{\text{Kr}}} \Delta_r^{(k)} K_k(r, r) \tag{67b}$$

so that, as we see, the Λ term coming from the LA now cancels the whole replicon contribution, not just the one with μ_{sing} .

The next item to be considered is a component of M with different upper indices, so that there is no replicon contribution to it:

$$\begin{aligned}
 M_{R+1,r}^{r,s} &= \sum_{k=0}^{R+1} \frac{\mu(k)}{n} \frac{(\delta_r - \frac{1}{2} p_{k-1} \delta_{r,k-1}^{Kr})(\delta_s - \frac{1}{2} p_{k-1} \delta_{s,k-1}^{Kr})}{\delta_r \delta_s} \frac{M_{r,s}^{(k)}}{g_s^{(k)}} \\
 &= \sum_{k=0}^{R+1} \frac{\mu(k)}{n} \frac{\Delta_r^{(k)} \Delta_s^{(k)}}{\delta_r \delta_s} K_k(r, s) \quad s \neq r.
 \end{aligned}
 \tag{68}$$

Now we turn to the matrix elements of the third kind:

$$\begin{aligned}
 M_{t,t}^{r,s} &= \sum_{k=0}^{R+1} \frac{2\mu(k)}{n \delta_s} \left(2 \frac{\Delta_t^{(k)}}{\delta_t} - 1 \right) M_{r,s}^{(k)} \\
 &= \sum_{k=0}^{R+1} \frac{\mu(k)}{n} K_k(r, s) \left(2 \frac{\Delta_t^{(k)}}{\delta_t} - 1 \right) \quad t < r, s.
 \end{aligned}
 \tag{69}$$

Note that this holds even for $r = s$, because

$$\sum_{k=0}^{R+1} \mu(k) \left(2 \frac{\Delta_t^{(k)}}{\delta_t} - 1 \right) = 0$$

therefore the diagonal part Λ in $M^{(k)}$ never contributes. Neither does the replicon, because the upper indices are larger than the lower ones in here.

The next four items are off-diagonal in the upper indices, so they do not receive contributions from the replicon family:

$$\begin{aligned}
 M_{r,t}^{r,s} &= \sum_{k=0}^{R+1} \frac{\mu(k)}{n} \frac{(\delta_s - \frac{1}{2} p_{k-1} \delta_{s,k-1}^{Kr})}{\delta_s} \left(2 \frac{\Delta_t^{(k)}}{\delta_t} - 1 \right) \frac{M_{r,s}^{(k)}}{g_s^{(k)}} \\
 &= \sum_{k=0}^{R+1} \frac{\mu(k)}{n} \frac{\Delta_s^{(k)}}{\delta_s} \left(2 \frac{\Delta_t^{(k)}}{\delta_t} - 1 \right) K_k(r, s) \quad s < t < r
 \end{aligned}
 \tag{70}$$

$$\begin{aligned}
 M_{r,t}^{r,s} &= \sum_{k=0}^{R+1} \frac{\mu(k)}{n} \frac{(\delta_r - \frac{1}{2} p_{k-1} \delta_{r,k-1}^{Kr})(\delta_s - \frac{1}{2} p_{k-1} \delta_{s,k-1}^{Kr})}{\delta_r \delta_s} \left(2 \frac{\Delta_t^{(k)}}{\delta_t} - 1 \right) \frac{M_{r,s}^{(k)}}{g_s^{(k)}} \\
 &= \sum_{k=0}^{R+1} \frac{\mu(k)}{n} \frac{\Delta_r^{(k)} \Delta_s^{(k)}}{\delta_r \delta_s} \left(2 \frac{\Delta_t^{(k)}}{\delta_t} - 1 \right) K_k(r, s) \quad r < s < t
 \end{aligned}
 \tag{71}$$

$$\begin{aligned}
 M_{r,r}^{r,s} &= \sum_{k=0}^{R+1} \frac{\mu(k)}{n} \left(2 \frac{\Delta_r^{(k)}}{\delta_r} - 1 \right) \frac{M_{r,s}^{(k)}}{g_s^{(k)}} \\
 &= \sum_{k=0}^{R+1} \frac{\mu(k)}{n} \left(2 \frac{\Delta_r^{(k)}}{\delta_r} - 1 \right) K_k(r, s) \quad r < s
 \end{aligned}
 \tag{72}$$

$$\begin{aligned}
 M_{r,s}^{r,s} &= \sum_{k=0}^{R+1} \frac{\mu(k)}{n} \frac{\delta_r - \frac{1}{2} p_{k-1} \delta_{r,k-1}^{Kr}}{\delta_r} \left(2 \frac{\Delta_s^{(k)}}{\delta_s} - 1 \right) \frac{M_{r,s}^{(k)}}{g_s^{(k)}} \\
 &= \sum_{k=0}^{R+1} \frac{\mu(k)}{n} \frac{\Delta_r^{(k)}}{\delta_r} \left(2 \frac{\Delta_s^{(k)}}{\delta_s} - 1 \right) K_k(r, s) \quad r < s.
 \end{aligned}
 \tag{73}$$

In the next two cases again a complete cancellation takes place between the Λ term in the LA and the replicon:

$$M_{r,r}^{r,r} = \sum_{k=0}^{R+1} \frac{2\mu(k)}{n\delta_r} \frac{\Delta_r^{(k)}}{\delta_r - \frac{1}{2}p_{k-1}\delta_{r,k-1}^{Kr}} \left(2\frac{\Delta_r^{(k)}}{\delta_r} - 1 \right) M_{r,r}^{(k)} + \frac{2\mu(r; r+1, r+1)}{n\delta_r} \frac{2p_{r+1}^2}{(p_r - 2p_{r+1})(p_r - 3p_{r+1})} \lambda(r; r+1, r+1) \quad (74a)$$

$$= \sum_{k=0}^{R+1} \frac{\mu(k)}{n} \left(2\frac{\Delta_r^{(k)}}{\delta_r} - 1 \right) K_k(r, r) \quad (74b)$$

$$M_{r,s}^{r,r} = \sum_{k=0}^{R+1} \frac{2\mu(k)}{n\delta_r} \frac{\Delta_r^{(k)}}{\delta_r - \frac{1}{2}p_{k-1}\delta_{r,k-1}^{Kr}} \left(\frac{\Delta_r^{(k)}}{\delta_r} + \frac{\Delta_s^{(k)}}{\delta_s} - 1 \right) \frac{\delta_r}{\Delta_r^{(k)}} M_{r,r}^{(k)} - \frac{p_{r+1}}{p_r - 2p_{r+1}} \left\{ \frac{2\mu(r; r+1, r+1)}{n\delta_r} \lambda(r; r+1, r+1) + \sum_{k=r+2}^{R+1} \frac{2\mu(r; r+1, k)}{n\delta_r} \lambda(r; r+1, k) \frac{\Delta_r^{(k)}\delta_s + \Delta_s^{(k)}\delta_r - \delta_r\delta_s}{\Delta_r^{(k)}\delta_s} \right\} \quad (75a)$$

$$= \sum_{k=0}^{R+1} \frac{\mu(k)}{n} \frac{\Delta_r^{(k)} - \frac{1}{4}p_{k-1}\delta_{r,k-1}^{Kr}}{\delta_r - \frac{1}{2}p_{k-1}\delta_{r,k-1}^{Kr}} \left(\frac{\Delta_r^{(k)}}{\delta_r} + \frac{\Delta_s^{(k)}}{\delta_s} - 1 \right) K_k(r, r) \quad s > r. \quad (75b)$$

Finally, in the last type of component the Λ term from the LA cancels the μ_{sing} term in the R:

$$M_{s,t}^{r,r} = \sum_{k=0}^{R+1} \frac{2\mu(k)}{n\delta_r} \left(\frac{\Delta_s^{(k)}}{\delta_s} + \frac{\Delta_t^{(k)}}{\delta_t} - 1 \right) M_{r,r}^{(k)} + \sum_{k=r+1}^{R+1} \sum_{l=r+1}^{R+1} \frac{2\mu(r; k, l)}{n\delta_r} \left(2\frac{\Delta_s^{(k)}}{\delta_s} - 1 \right) \left(2\frac{\Delta_t^{(l)}}{\delta_t} - 1 \right) \lambda(r; k, l) \quad (76a)$$

$$= \sum_{k=0}^{R+1} \frac{\mu(k)}{n\delta_r} \Delta_r^{(k)} \left(\frac{\Delta_s^{(k)}}{\delta_s} + \frac{\Delta_t^{(k)}}{\delta_t} - 1 \right) K_k(r, r) + \sum_{k=r+1}^{R+1} \sum_{l=r+1}^{R+1} \frac{2\mu_{\text{reg}}(r; k, l)}{n\delta_r} \left(2\frac{\Delta_s^{(k)}}{\delta_s} - 1 \right) \left(2\frac{\Delta_t^{(l)}}{\delta_t} - 1 \right) \lambda(r; k, l) \quad s, t > r. \quad (76b)$$

As we have mentioned, in (65)–(76) the formulae denoted (a) give the true partition of the contributions between the LA and R families. If anyone tried to reproduce these results, they would inevitably get them in this form, and we give them here partly as signposts. In most applications the origin of the terms is completely immaterial, however, so that when using these formulae, one will clearly apply the (b) forms, where the cancellations have been performed. The names one attaches to these terms are also largely a matter of convention: in the papers [14, 15], where analogous formulae were derived for the propagators, two of us used the name LA for the first terms and the name R for the second terms in the (b) forms.

We also see that there is nothing mysterious about the cancellations: the diagonal matrix elements of $M^{(k)}$ contain the replicon eigenvalue and this piece partially or completely

cancels the contribution from the R family. In [13] two of us, discussing the importance of these cancellations in the context of the propagators, made the remark that a certain asymptotic relation between the second and third family eigenvalues was a necessary condition for the cancellations to work. Although the asymptotic relation between the eigenvalues was certainly valid in the specific example discussed there, and may also be valid in more general situations, we can clearly see that it has nothing to do with the cancellations: these are a purely 'kinematic' effect, depending solely on the ultrametric geometry and on no further details of the theory.

To conclude, we make an additional remark. Before presenting the formulae (65)–(76) we gave a sketchy indication (with some details to be added in the appendix) as to how they can be obtained, which is, of course, not necessarily the most economic way that they can be verified once known. Equations (65)–(76) are the inversion of (47)–(52) and of (41). The simplest way to check them is by direct substitution.

Having established the inverse relations between the matrix elements and the kernel we can now summarize the steps one has to follow in order to invert an ultrametric matrix. First one has to determine the kernel and the replicon eigenvalues of the matrix by (41) and (47)–(52). To get the replicon eigenvalues of the inverse matrix is trivial; they are the reciprocal of the original replicon eigenvalues. To obtain the new kernel requires the solution of (57). This is the hard core which remains to be cracked after the layer controlled by ultrametricity has been peeled off. To find the new kernel requires concrete knowledge of the matrix elements and as such it is outside the scope of the present paper. Assuming the new kernel has been found, one finally obtains the elements of the inverse matrix via (65)–(76).

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Appendix.

Our purpose here is to sketch the derivation of the replicon contributions to (65)–(76) through what we called the representative vector.

Equation (58), written out in the original, Cartesian coordinates, reads

$$M_{\alpha\beta,\gamma\delta} = \sum_{ij} \langle \alpha\beta | i \rangle \tilde{M}_{ij} \langle \tilde{j} | \gamma\delta \rangle \quad (\text{A1})$$

where $|i\rangle$ now means any of the $\frac{1}{2}n(n-1)$ new basis vectors, and $\langle \tilde{j} |$ are their bi-orthogonal counterparts.

We are interested here in the contribution of the replicon family to (A1) only, i.e. in the partial sum, to be denoted by $M_{\alpha\beta,\gamma\delta}^R$, where i and j are restricted to the replicon sector. But \tilde{M} is diagonal in that sector, $\tilde{M}_{ij} = \lambda_i \delta_{i,j}^{Kr}$, so we have

$$M_{\alpha\beta,\gamma\delta}^R = \sum_{i \in R} \langle \alpha\beta | i \rangle \lambda_i \langle \tilde{i} | \gamma\delta \rangle. \quad (\text{A2})$$

Now let $\alpha \cap \beta = r$. Then the replicon vectors $|i\rangle$ contributing to (A2) have to be such that they have non-zero components on the r th level of the Parisi hierarchy. From the description of these vectors given in the main text we know, however, that their components on every other level are then identically zero, and furthermore even on the r th level their non-zero components are concentrated inside a single block of size $p_r \times p_r$. Evidently, the component $(\alpha\beta)$ must belong to this block. Although we have not actually determined the bi-orthogonal set $\langle i|$ (and our purpose here is to show that we do not need to, either), it is obvious that $\langle i|$ will share the above properties of $|i\rangle$: it will have non-zero components on the r th level, and there inside the same $p_r \times p_r$ block only. It follows that $(\gamma\delta)$ must belong to the same block and $\alpha \cap \beta = \gamma \cap \delta = r$.

With this we have identified the set of replicon vectors which, for a given $\alpha, \beta, \gamma, \delta$, can give a non-zero contribution to $M_{\alpha\beta, \gamma\delta}^R$. This set can be decomposed into orthogonal classes, labelled by the triplet of integers r, k, l ($k, l \geq r + 1$), as explained in section 3. In a given class (r, k, l) we still have several non-orthogonal replicon vectors and, in principle, they all contribute to (A2). What we wish to show is that, in fact, one can choose a single vector from each class (r, k, l) in such a way as to exhaust the contribution of the whole class. We have called this vector the representative vector of the class.

The choice of the representative vector is not unique. It is best to choose it such that the component $(\gamma\delta)$ belong to the 'darkest' block, where the vector has the components A (consult figures 5–7). Now, as we have already pointed out in section 3, the subspace orthogonal to the representative vector thus chosen is spanned by vectors that have zero components over this 'A-block'. In particular, if $|r; k, l\rangle$ is the representative vector of the class (r, k, l) then the bi-orthogonal counterparts of all the other members of the class will lie in the space orthogonal to $|r; k, l\rangle$, hence their components in the 'A-block' where also the pair $(\gamma\delta)$ resides must necessarily be zero and thus their scalar product with the unit vector $|\gamma\delta\rangle$ will vanish. Therefore, the only contribution from the class (r, k, l) comes from the representative vector, indeed.

The summation in (A2) then runs over the representative vectors only, so we can rewrite (A2) as

$$M_{\alpha\beta, \gamma\delta}^R = \sum_{k=r+1}^{R+1} \sum_{l=r+1}^{R+1} \lambda(r; k, l) (\alpha\beta|r; k, l) \widetilde{\langle r; k, l|\gamma\delta\rangle}. \quad (\text{A3})$$

We now decompose $\widetilde{\langle r; k, l|}$ into components parallel and orthogonal to $|r; k, l\rangle$. The orthogonal component will not contribute to (A3) for the same reasons as above. So we are left with the parallel component only, which, in view of $\widetilde{\langle r; k, l|r; k, l\rangle} = 1$ and of the normalization of $|r; k, l\rangle$, is nothing but the representative vector itself. So we can finally write

$$M_{\alpha\beta, \gamma\delta}^R = \sum_{k=r+1}^{R+1} \sum_{l=r+1}^{R+1} \lambda(r; k, l) (\alpha\beta|r; k, l) \langle r; k, l|\gamma\delta\rangle. \quad (\text{A4})$$

Although the above consideration is quite trivial really, one may find it mystifying that it is possible to reconstruct a matrix from selecting a single vector from each class of its eigenvectors which are non-orthogonal within the class. As it transpires from the proof, the key factor is that the subspace orthogonal to the representative vector is composed of vectors having vanishing components over the 'A-block', and this in turn hinges upon the common property of all replicons, namely that the sum of their components in each row

is zero. A little more reflection will show, however, that the underlying reason is that the matrix $M_{\alpha\beta,\gamma\delta}^R$ does not depend on α , β , γ and δ separately, only on the various overlaps formed out of these indices, therefore the puzzling property of the representative vectors carrying all the information about M^R can be directly linked to the ultrametric symmetries of M .

As an illustration of the use of (A4), let us calculate the diagonal components $M_{\alpha\beta,\alpha\beta}^R$. There will be three kinds of terms contributing to (A4):

(i) $k = l = r + 1$:

$$\langle r; r+1, r+1 | \alpha\beta \rangle^2 = A^2 = \frac{p_r - 3p_{r+1}}{p_{r+1}^2(p_r - p_{r+1})} = \frac{2\mu(r; r+1, r+1)}{n\delta_r} \quad (\text{A5})$$

where use has been made of (32) and (33).

(ii) $k \geq r + 2, l = r + 1$:

$$\langle r; k, r+1 | \alpha\beta \rangle^2 = A^2 = \frac{p_r - 2p_{r+1}}{p_{r+1}(p_r - p_{r+1})} \left(\frac{1}{p_k} - \frac{1}{p_{k-1}} \right) = \frac{2\mu(r; k, r+1)}{n\delta_r} \quad (\text{A6})$$

where we have used (34) and (35).

(iii) $k \geq r + 2, l \geq r + 2$:

$$\langle r; k, l | \alpha\beta \rangle^2 = A^2 = \left(\frac{1}{p_k} - \frac{1}{p_{k-1}} \right) \left(\frac{1}{p_l} - \frac{1}{p_{l-1}} \right) = \frac{2\mu(r; k, l)}{n\delta_r} \quad (\text{A7})$$

see (36) and (37).

Substituting (A5), (A6) and (A7) back into (A4) we find

$$M_{\alpha\beta,\alpha\beta}^R = (M^R)_{R+1,R+1}^{r,r} = \sum_{k=r+1}^{R+1} \sum_{l=r+1}^{R+1} \frac{2\mu(r; k, l)}{n\delta_r} \lambda_{(r; k, l)}$$

which is precisely the second (replicon) term quoted in (65a). The replicon contributions to (66a), (67a), (74a), (75a) and (76a) can be worked out similarly.

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